Optimal and Learning Control
for
Autonomous Robots
Lecture 6

A D R L

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Erratum Script

p14

$$\frac{dV^*}{dt} = V_t^* + V_{\mathbf{x}}^{*T}\mathbf{f} + \frac{1}{2}\mathrm{Tr}\left[V_{\mathbf{xx}}^* E[(\mathbf{f}+\mathbf{B}\mathbf{w})(\mathbf{f}+\mathbf{B}\mathbf{w})^T]\Delta t\right]. \quad (1.55)$$

p28

$$-\mathbf{x}^T\dot{\mathbf{S}}(t)\mathbf{x} = \min_{u\in U}\{\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u} + 2\mathbf{u}^T\mathbf{P}\mathbf{x} + 2\mathbf{x}^T\mathbf{S}(t)\mathbf{A}\mathbf{x} + 2\mathbf{x}^T\mathbf{S}(t)\mathbf{B}\mathbf{u}\}. \quad (1.105)$$

Equations (1.27), (1.28), (1.29):

$$V^\mu(n,\mathbf{x}) = L_n(\mathbf{x},\mathbf{u}_n) + \underline{\alpha} E_{x'\sim P_f(.|\mathbf{x},\mathbf{u}_n)}\left[V^\mu\left(n+1,\mathbf{x}'\right)\right] \quad (1.27)$$

$$V^*(n,\mathbf{x}) = \min_{\mathbf{u}}\left[L_n(\mathbf{x},\mathbf{u}_n) + \underline{\alpha} E_{\mathbf{x}'\sim P_f(\cdot|\mathbf{x},\mathbf{u}_n)}\left[V^*\left(n+1,\mathbf{x}'\right)\right]\right] \quad (1.28)$$

$$\mathbf{u}^*(n,\mathbf{x}) = \arg\min_{\mathbf{u}_n}\left[L_n(\mathbf{x},\mathbf{u}_n) + \underline{\alpha} E_{\mathbf{x}'\sim P_f(\cdot|\mathbf{x},\mathbf{u}_n)}\left[V^*\left(n+1,\mathbf{x}'\right)\right]\right]. \quad (1.29)$$

Equations (1.143), (1.145):

$$\mathbf{u}^*(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{P} + \mathbf{B}^T\underline{\mathbf{S}})\mathbf{x} \quad (1.143)$$

$$\mathbf{u}^*(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{P} + \mathbf{B}^T\underline{\mathbf{S}})\mathbf{x} \quad (1.145)$$

Class logistics

Exercise 1

- Submission:
  - Code must be submitted through website form
  - NO EMAIL SUBMISSION!
  - submit by Wed, 15.4.2015
  - USE OFFICE HOURS FOR QUESTIONS!

Office hours:
Thu, 17:30-18:30
Room: ML J37.1

- Interviews:
  - Interviews on Friday, 17.4.2015, all day
  - 10 min session/group
  - explain submitted code and answers
  - pass/fail grade given
  - Doodle link for sign up for interview will be given

Lecture 6 Goals

★ LQG - Linear quadratic regulator with gaussian noise

L5 Recap

Solve optimal control problem

$$V^*(n, \mathbf{x}) = \min_{\mathbf{u}_n} \left[L_n(\mathbf{x}, \mathbf{u}_n) + \alpha V^* \left(n+1, \mathbf{f}_n \left(\mathbf{x}, \mathbf{u}_n \right) \right) \right]$$

1. Principle of optimality: Bellman / HJB Equation
2. Make some assumptions
3. Minimize RHS of Equation
4. ... yields conditions for optimal control
5. substitute back to solve for remaining quantities

SQP vs SLQ

SQP

1. Initial guess for parameter
2. Solve sub problem: Approximate original problem with a linear-quadratic problem
3. yields new approximative solution
4. repeat

SLQ

1. Initial guess for policy
2. Solve sub problem: Approximate value function with a linear-quadratic
3. yields new approximative policy
4. repeat

SLQ subproblem in a nutshell

2.1 Forward pass:

integrate to get a state (and controls) trajectory

2.2 Backward pass

Solve simplified optimal control problem around state and control trajectory

3. Adjust guess for optimal control

choice of: approximation, solver ⇒different SLQ algorithms

(examples: DDP, iLQG, ILQC)

ILQC
Overview of derivation

Linearize system dynamics
Quadratize cost

Compute value function
Compute optimal control
Solve for Riccati like equation
Solve Riccati like equation

ILQC main iteration

0. *Initialization:* we assume that an initial, feasible policy $\boldsymbol{\mu}$ and initial state $\mathbf{x}_0$ is given. Then, for every iteration (i):

nonlinear system

$$\bar{\mathbf{x}}_{n+1} = \mathbf{f}_n(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)$$

Forward pass

1. *Roll-Out:* perform a forward-integration of the system dynamics (1.70) subject to initial condition $\mathbf{x}_0$ and the current policy $\boldsymbol{\mu}$. Thus, obtain the nominal state- and control input trajectories $\bar{\mathbf{u}}_n^{(i)}, \bar{\mathbf{x}}_n^{(i)}$ for $n = 0, 1, \ldots, N$.

$$\delta\mathbf{x}_{n+1} \approx \mathbf{A}_n\delta\mathbf{x}_n + \mathbf{B}_n\delta\mathbf{u}_n$$

$$\mathbf{A}_n = \frac{\partial \mathbf{f}(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)}{\partial \mathbf{x}}$$

$$\mathbf{B}_n = \frac{\partial \mathbf{f}(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)}{\partial \mathbf{u}}$$

2. *Linear-Quadratic Approximation:* build a local, linear-quadratic approximation around every state-input pair $(\bar{\mathbf{u}}_n^{(i)}, \bar{\mathbf{x}}_n^{(i)})$ as described in Equations (1.75) to (1.78).

Backwards pass

3. *Compute the Control Law:* solve equations (1.84) to (1.86) backward in time and design the affine control policy through equation (1.88).

$$\mathbf{u}(n, x) = \bar{\mathbf{u}}_n + \delta\mathbf{u}_n^{ff} + \mathbf{K}_n(\mathbf{x}_n - \bar{\mathbf{x}}_n)$$

4. Go back to 1. and repeat until the sequences $\bar{\mathbf{u}}^{(i+1)}$ and $\bar{\mathbf{u}}^{(i)}$ are sufficiently close.

$$\mathbf{S}_n = \mathbf{Q}_n + \mathbf{A}_n^T\mathbf{S}_{n+1}\mathbf{A}_n + \mathbf{K}_n^T\mathbf{H}_n\mathbf{K}_n + \mathbf{K}_n^T\mathbf{G}_n + \mathbf{G}_n^T\mathbf{K}_n$$

$$\mathbf{s}_n = \mathbf{q}_n + \mathbf{A}_n^T\mathbf{s}_{n+1} + \mathbf{K}_n^T\mathbf{H}_n\delta\mathbf{u}_n^{ff} + \mathbf{K}_n^T\mathbf{g}_n + \mathbf{G}_n^T\delta\mathbf{u}_n^{ff}$$

$$s_n = q_n + s_{n+1} + \frac{1}{2}\delta\mathbf{u}_\mathbf{n}^{\mathbf{ff}\,T}\mathbf{H}_n\delta\mathbf{u}_n^{ff} + \delta\mathbf{u}_n^{ff\,T}\mathbf{g}_n$$

$$J \approx q_N + \delta\mathbf{x}_N^T\mathbf{q}_N + \frac{1}{2}\delta\mathbf{x}_N^T\mathbf{Q}_N\delta\mathbf{x}_N + \sum_{n=0}^{N-1}\{q_n + \delta\mathbf{x}_n^T\mathbf{q}_n + \delta\mathbf{u}_n^T\mathbf{r}_n + \frac{1}{2}\delta\mathbf{x}_n^T\mathbf{Q}_n\delta\mathbf{x}_n + \frac{1}{2}\delta\mathbf{u}_n^T\mathbf{R}_n\delta\mathbf{u}_n + \delta\mathbf{u}_n^T\mathbf{P}_n\delta\mathbf{x}_n\}$$

$\forall n \in \{0, \cdots, N-1\}$:

$$q_n = L_n(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n), \quad \mathbf{q}_n = \frac{\partial L(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)}{\partial \mathbf{x}}, \quad \mathbf{Q}_n = \frac{\partial^2 L(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)}{\partial \mathbf{x}^2}$$

$$\mathbf{P}_n = \frac{\partial^2 L(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)}{\partial \mathbf{u}\partial \mathbf{x}}, \quad \mathbf{r}_n = \frac{\partial L(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)}{\partial \mathbf{u}}, \quad \mathbf{R}_n = \frac{\partial^2 L(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)}{\partial \mathbf{u}^2}$$

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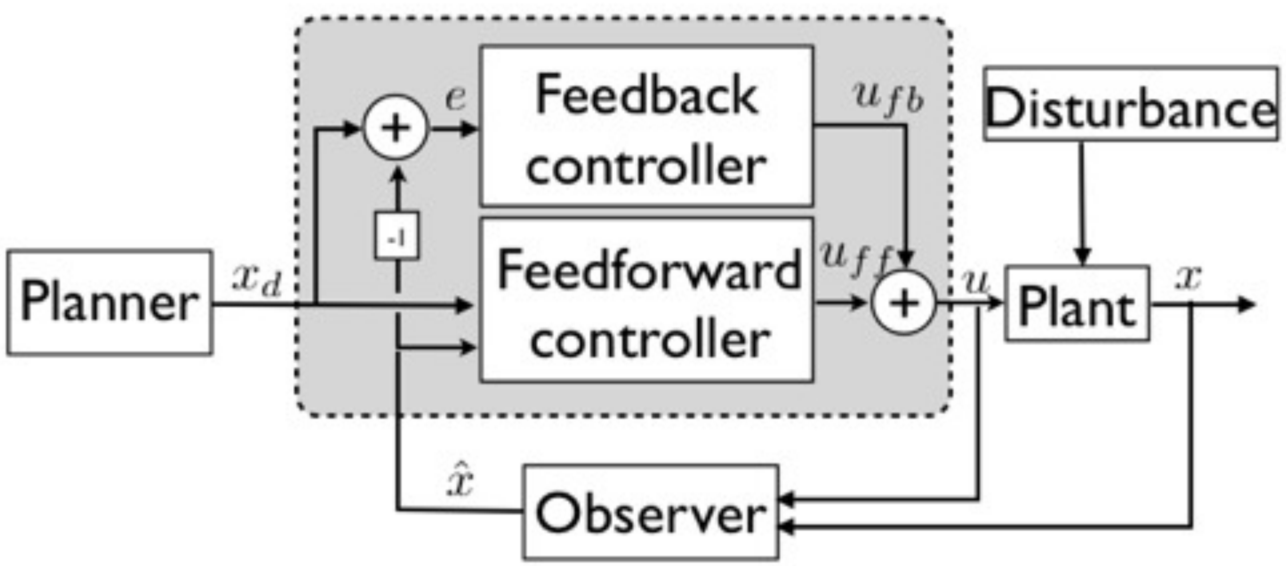
Optimal control: FF/FB

$$\delta\mathbf{u}_n = -\mathbf{H}_n^{-1}\mathbf{g}_n - \mathbf{H}_n^{-1}\mathbf{G}_n\delta\mathbf{x}_n$$

feed-forward term $\delta\mathbf{u}_n^{ff} = -\mathbf{H}_n^{-1}\mathbf{g}_n$

feedback term $\mathbf{K}_n\delta\mathbf{x}_n$ feedback gain matrix $\mathbf{K}_n := -\mathbf{H}_n^{-1}\mathbf{G}_n$

$$\delta\mathbf{u}_n = \delta\mathbf{u}_n^{ff} + \mathbf{K}_n\delta\mathbf{x}_n$$



LQR - Linear Quadratic Regulator

Linearized System Dynamics

Quadratic cost function

Regulates output to zero

Discrete Time LQR

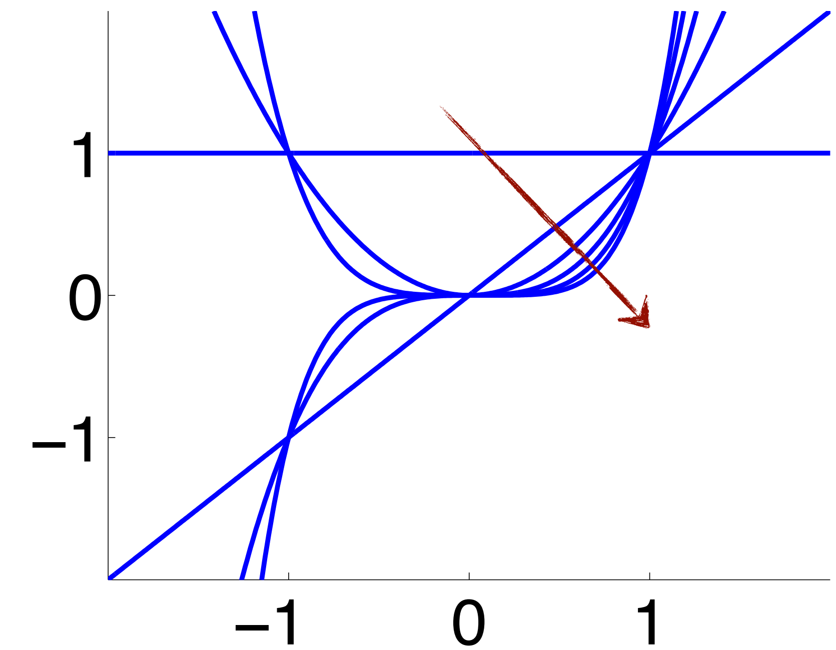
Local approximations...

Taylor series are polynomials

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i$$

polynomials can (locally) approximate arbitrary function

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$



$x^0 \quad x^1 \quad x^2 \quad x^3 \quad x^4 \quad x^5$

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Optimal Regulator

Linear

- ★ Control:
  pure state feedback (no forward)
  linear control enough to stabilize locally

Quadratic

- ★ Cost:
  regulator: optimum at x,u = 0
  increasing for any non-zero x,u
  ⇒ purely quadratic cost

Quadratization of cost function

$$J = \Phi(\mathbf{x}_N) + \sum_{n=0}^{N-1} L_n(\mathbf{x}_n, \mathbf{u}_n)$$

$$J \approx q_N + \delta\mathbf{x}_N^T \mathbf{q}_N + \frac{1}{2}\delta\mathbf{x}_N^T \mathbf{Q}_N \delta\mathbf{x}_N$$
$$+ \sum_{n=0}^{N-1} \{q_n + \delta\mathbf{x}_n^T \mathbf{q}_n + \delta\mathbf{u}_n^T \mathbf{r}_n + \frac{1}{2}\delta\mathbf{x}_n^T \mathbf{Q}_n \delta\mathbf{x}_n + \frac{1}{2}\delta\mathbf{u}_n^T \mathbf{R}_n \delta\mathbf{u}_n + \delta\mathbf{u}_n^T \mathbf{P}_n \delta\mathbf{x}_n\}$$

Control costs

State costs

'Mixing terms'

$\forall n \in \{0, \cdots, N-1\}:$

$$q_n = L(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n), \quad \mathbf{q}_n = \frac{\partial L(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)}{\partial \mathbf{x}}, \quad \mathbf{Q}_n = \frac{\partial^2 L(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)}{\partial \mathbf{x}^2}$$

$$\mathbf{P}_n = \frac{\partial^2 L(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)}{\partial \mathbf{u} \partial \mathbf{x}}, \quad \mathbf{r}_n = \frac{\partial L(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)}{\partial \mathbf{u}}, \quad \mathbf{R}_n = \frac{\partial^2 L(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)}{\partial \mathbf{u}^2}$$

$n = N:$

$$q_N = \Phi(\bar{\mathbf{x}}_n), \quad \mathbf{q}_N = \frac{\partial \Phi(\bar{\mathbf{x}}_n)}{\partial \mathbf{x}}, \quad \mathbf{Q}_N = \frac{\partial^2 \Phi(\bar{\mathbf{x}}_n)}{\partial \mathbf{x}^2}$$

Note that all derivatives w.r.t. $\mathbf{u}$ are zero for the terminal time-step N

Q, R, P are given through definition of cost!

$$q_n = L_n(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n) = 0 \qquad \mathbf{q}_n = \frac{\partial L(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)}{\partial \mathbf{x}} = \mathbf{0} \qquad \mathbf{r}_n = \frac{\partial L(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)}{\partial \mathbf{u}} = \mathbf{0}$$

$$\mathbf{S}_N = \mathbf{Q}_N, \qquad \mathbf{s}_N = \mathbf{0}, \qquad s_N = \mathbf{0}$$

$$\mathbf{g}_n \triangleq \mathbf{r}_n + \mathbf{0}\,\mathbf{s}_{n+1}$$
$$\mathbf{G}_n \triangleq \mathbf{P}_n + \mathbf{B}_n^T \mathbf{S}_{n+1} \mathbf{A}_n$$
$$\mathbf{H}_n \triangleq \mathbf{R}_n + \mathbf{B}_n^T \mathbf{S}_{n+1} \mathbf{B}_n$$

$$\delta \mathbf{u}_n^{ff} = -\mathbf{H}\,\mathbf{0}\,\mathbf{g}_n$$

$$\mathbf{S}_n = \mathbf{Q}_n + \mathbf{A}_n^T \mathbf{S}_{n+1} \mathbf{A}_n + \mathbf{K}_n^T \mathbf{H}_n \mathbf{K}_n + \mathbf{K}_n^T \mathbf{G}_n + \mathbf{G}_n^T \mathbf{K}_n$$
$$\mathbf{s}_n = \mathbf{q}_n + \mathbf{A}_n^T \mathbf{s}_{n+1} + \mathbf{K}_n^T \mathbf{H}_n \mathbf{0}\mathbf{u}_n^{ff} + \mathbf{K}_n^T \mathbf{g}_n + \mathbf{G}_n^T \delta \mathbf{u}_n^{ff}$$
$$s_n = q_n + s_{n+1} + \frac{1}{2} \delta \mathbf{u}_n^{ff^T} \mathbf{H}_n \mathbf{0}\mathbf{u}_n^{ff} + \delta \mathbf{u}_n^{ff^T} \mathbf{g}_n$$

Purely Quadratic cost

$$J = \frac{1}{2}\mathbf{x}_N^T \mathbf{Q}_N \mathbf{x}_N + \sum_{n=0}^{N-1} \frac{1}{2}\mathbf{x}_n^T \mathbf{Q}_n \mathbf{x}_n + \frac{1}{2}\mathbf{u}_n^T \mathbf{R}_n \mathbf{u}_n + \mathbf{u}_n^T \mathbf{P}_n \mathbf{x}_n$$

at optimum no linear term (locally symmetric)

cf with polynomial

$$q_n = L_n(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n) = 0 \qquad \mathbf{q}_n = \frac{\partial L(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)}{\partial \mathbf{x}} = \mathbf{0} \qquad \mathbf{r}_n = \frac{\partial L(\bar{\mathbf{x}}_n, \bar{\mathbf{u}}_n)}{\partial \mathbf{u}} = \mathbf{0}$$

Ansatz for Value Function

$$V^*(n+1, \delta\mathbf{x}_{n+1}) = s_{n+1} + \delta\mathbf{x}_{n+1}^T\mathbf{s}_{n+1} + \frac{1}{2}\delta\mathbf{x}_{n+1}^T\mathbf{S}_{n+1}\delta\mathbf{x}_{n+1}$$

$$V^*(n, \mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{S}_n\mathbf{x}$$

Ricatti Equation

$$\mathbf{K}_n := -\mathbf{H}_n^{-1}\mathbf{G}_n$$

$$\mathbf{S}_n = \mathbf{Q}_n + \mathbf{A}_n^T\mathbf{S}_{n+1}\mathbf{A}_n + \mathbf{K}_n^T\mathbf{H}_n\mathbf{K}_n + \mathbf{K}_n^T\mathbf{G}_n + \mathbf{G}_n^T\mathbf{K}_n$$
$$= \mathbf{Q}_n + \mathbf{A}_n^T\mathbf{S}_{n+1}\mathbf{A}_n - \mathbf{G}_n^T\mathbf{H}_n^{-1}\mathbf{G}_n$$

$$\mathbf{G}_n \triangleq \mathbf{P}_n + \mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{A}_n$$
$$\mathbf{H}_n \triangleq \mathbf{R}_n + \mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{B}_n$$

Discrete time Riccati equation

$$\mathbf{S}_n = \mathbf{Q}_n + \mathbf{A}_n^T\mathbf{S}_{n+1}\mathbf{A}_n - (\mathbf{P}_n^T + \mathbf{A}_n^T\mathbf{S}_{n+1}\mathbf{B}_n)(\mathbf{R}_n + \mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{B}_n)^{-1}(\mathbf{P}_n + \mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{A}_n)$$

solve backwards $\mathbf{S}_N = \mathbf{Q}_N$

Optimal policy

$$\boldsymbol{\mu}^*(n, \mathbf{x}) = -\mathbf{H}_n^{-1}\mathbf{G}_n\mathbf{x}$$
$$= -(\mathbf{R}_n + \mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{B}_n)^{-1}(\mathbf{P}_n + \mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{A}_n)\mathbf{x}$$

Pure feedback / no feedforward

Continuous time LQR

Continuos-time linear time variant system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

$$J = \frac{1}{2}\mathbf{x}(T)^T\mathbf{Q}_T\mathbf{x}(T) + \int_0^T \left(\frac{1}{2}\mathbf{x}(t)^T\mathbf{Q}(t)\mathbf{x}(t) + \frac{1}{2}\mathbf{u}(t)^T\mathbf{R}(t)\mathbf{u}(t) + \mathbf{u}(t)^T\mathbf{P}(t)\mathbf{x}(t)\right) dt.$$

Hamilton Jacobi Bellman Equation:

$$-\frac{\partial V^*}{\partial t} = \min_{u \in U}\{L(x,u) + \left(\frac{\partial V^*}{\partial x}\right)^T f(x,u)\}$$

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Riccati Equation

Optimal control

$$\mathbf{u}^*(t,\mathbf{x}) = -\mathbf{R}^{-1}\left(\mathbf{P} + \mathbf{B}^T\mathbf{S}(t)\right)\mathbf{x}$$

$$\dot{\mathbf{S}} = -\mathbf{S}\mathbf{A} - \mathbf{A}^T\mathbf{S} + \left(\mathbf{P} + \mathbf{B}^T\mathbf{S}\right)^T\mathbf{R}^{-1}\left(\mathbf{P} + \mathbf{B}^T\mathbf{S}\right) - \mathbf{Q}$$

$$\mathbf{S}(T) = \mathbf{Q}_T$$

EOF Recap

L6

Stochastic LQR?!

LQG(R)

LQR + additive gaussian noise

Stochasticity where?

5.2 NONLINEAR SYSTEMS WITH RANDOM INPUTS AND IMPERFECT MEASUREMENTS

If a dynamic system is driven by uncertain disturbances, the stochastic optimal trajectory must be generated using measurements of the state. If the measurements contain random errors, then the feedback control responds not only to the effects of random inputs, but to the measurement errors as well. The control forces resulting from measurement errors are, of course, spurious, so it is desirable to minimize their effects, at the same time transmitting the maximum amount of information about the state. Therefore, the best neighboring control strategy involves optimal state estimation as well as optimal control. The estimator introduces *caution* or *hedging* in the feedback control by not responding to deviations that probably are due to measurement error.

from [Stengel, "Optimal control & Estimation", 1994]

Optimal Estimation

We will not look at the optimal estimation problem!

BUT: It can be treated with the same tools (c.f. Kalman Filter)

See: Certainty Equivalence Property

LQG(R) – Linear Quadratic Gaussian Regulator

Linear system dynamics

Quadratic cost function

Gaussian process noise

Regulates states to zero

LQG Problem

- It is a stochastic problem.
- The LQR-type cost function will be stochastic
- LQG cost function is expectation of LQR cost function
- The noise is Gaussian, not any arbitrary noise
  - Gaussian noise have physical interpretations.
  - Most importantly, it has a nice analytical feature: It is closed under a linear transformation.

Discrete Time LQG: Finite Horizon

Problem Definition

- Linear system dynamics contaminated by a Gaussian noise

$$\mathbf{x}_{n+1} = \mathbf{A}_n \mathbf{x}_n + \mathbf{B}_n \mathbf{u}_n + \mathbf{C}_n \mathbf{w}_n \qquad \mathbf{x}(0) = \mathbf{x}_0 \text{ given}$$

Zero mean

$$E[\mathbf{w}_n] = 0$$

$$E[\mathbf{w}_n \mathbf{w}_m^T] = \mathbf{I}\delta(n-m)$$

Uncorrelated

Gaussian process should be defined by its **Mean** & **Covariance**

ADRL

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Problem Definition

- Linear system dynamics contaminated by a Gaussian noise

$$\mathbf{x}_{n+1} = \mathbf{A}_n \mathbf{x}_n + \mathbf{B}_n \mathbf{u}_n + \mathbf{C}_n \mathbf{w}_n \qquad \mathbf{x}(0) = \mathbf{x}_0 \text{ given}$$

$$E[\mathbf{w}_n] = 0$$

$$E[\mathbf{w}_n \mathbf{w}_m^T] = \mathbf{I}\delta(n-m)$$

How to simulate it:
In each time step, extract a sample from a Gaussian distribution with zero mean and identity covariance.

Problem Definition

- Linear system dynamics contaminated by a Gaussian noise

$$\mathbf{x}_{n+1} = \mathbf{A}_n \mathbf{x}_n + \mathbf{B}_n \mathbf{u}_n + \mathbf{C}_n \mathbf{w}_n \qquad \mathbf{x}(0) = \mathbf{x}_0 \text{ given}$$

Can be any arbitrary functions of time.

However they should be bounded.

Problem Definition

- Linear system dynamics contaminated by a Gaussian noise

$$\mathbf{x}_{n+1} = \mathbf{A}_n \mathbf{x}_n + \mathbf{B}_n \mathbf{u}_n + \underbrace{\mathbf{C}_n \mathbf{w}_n} \qquad \mathbf{x}(0) = \mathbf{x}_0 \text{ given}$$

It is a Gaussian process:

$$E[\mathbf{C}_n \mathbf{w}_n] = \mathbf{C}_n E[\mathbf{w}_n] = 0$$

$$E\left[(\mathbf{C}_n \mathbf{w}_n)(\mathbf{C}_m \mathbf{w}_m)^T\right] = \mathbf{C}_n E\left[\mathbf{w}_n \mathbf{w}_m^T\right] \mathbf{C}_m^T = \mathbf{C}_n \mathbf{C}_m^T \delta(n-m)$$

Problem Definition

- Pure quadratic cost function

$$J = \frac{1}{2} E \left\{ \alpha^N \mathbf{x}_N^T \mathbf{Q}_N \mathbf{x}_N + \sum_{n=0}^{N-1} \alpha^n \begin{bmatrix} \mathbf{x}_n^T & \mathbf{u}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{Q}_n & \mathbf{P}_n^T \\ \mathbf{P}_n & \mathbf{R}_n \end{bmatrix} \begin{bmatrix} \mathbf{x}_n \\ \mathbf{u}_n \end{bmatrix} \right\}$$

Without *E* and α it is equivalent to the LQR cost function

It is Just written in vector form

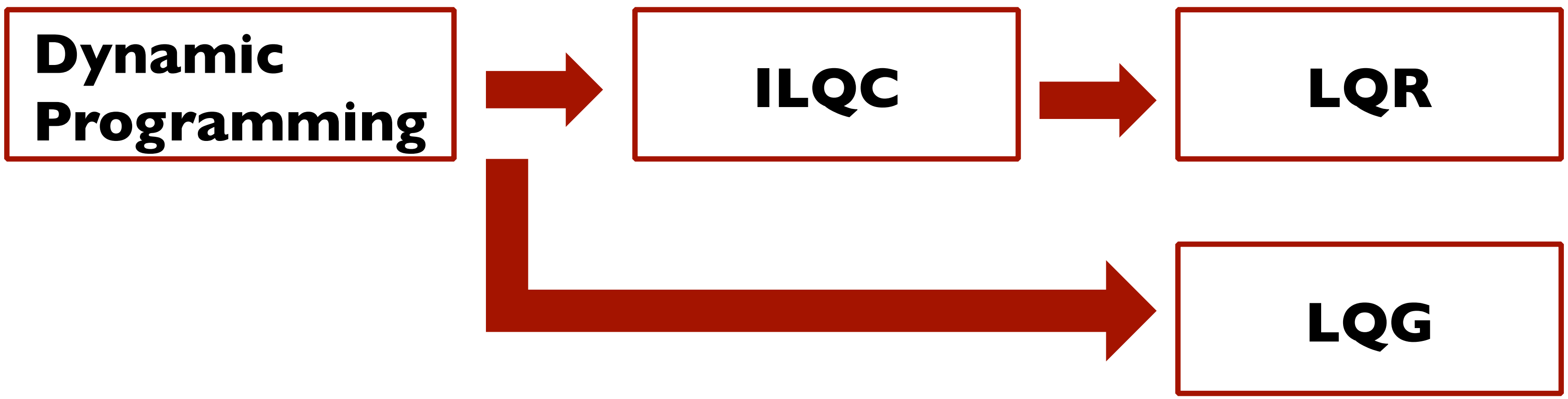
Problem Definition

- Pure quadratic cost function

$$J = \frac{1}{2} E \left\{ \alpha^N \mathbf{x}_N^T \mathbf{Q}_N \mathbf{x}_N + \sum_{n=0}^{N-1} \alpha^n \begin{bmatrix} \mathbf{x}_n^T & \mathbf{u}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{Q}_n & \mathbf{P}_n^T \\ \mathbf{P}_n & \mathbf{R}_n \end{bmatrix} \begin{bmatrix} \mathbf{x}_n \\ \mathbf{u}_n \end{bmatrix} \right\}$$

- α is discount factor or decay factor.
- For finite horizon it is usually chosen 1.
- However for infinite horizon it should be chosen smaller than 1; otherwise the summation does not exist.

How to Solve it?



We just need to choose the correct formula!

How to Solve it?

	Discrete Time	Continuous Time
Stochastic System	Optimization Problem: $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, \mathbf{u}_n) + \mathbf{w}_n$ $\mathbf{w}_n \sim P_{\mathbf{w}}(\cdot \mid \mathbf{x}_n, \mathbf{u}_n)$ $\min_{\mathbf{u}_{0\to N-1}} E\{\alpha^N \Phi(N) + \sum_{k=0}^{N-1} \alpha^k L(\mathbf{x}_k, \mathbf{u}_k)\} \quad \alpha \in [0,1]$ Stochastic Bellman equation: $V^*(n,\mathbf{x}) = \min_{u_n}\{L(x, u_n)$ $+ \alpha E[V^*(n+1, \mathbf{x}_{n+1})]\}$ Infinite horizon: $\alpha \in [0,1)$ $\Phi(N) = 0$ V^* is not function of time.	Optimization Problem: $d\mathbf{x} = \mathbf{f}(\mathbf{x}_t, \mathbf{u}_t)dt + \mathbf{B}(\mathbf{x}_t, \mathbf{u}_t)d\mathbf{w}_t$ $\mathbf{w}_t \sim \mathcal{N}(0, \Sigma)$ $\min_{\mathbf{u}_{0\to t_f}} E\{e^{-\beta t_f}\Phi(t_f) + \int_0^{t_f} e^{-\beta t} L(\mathbf{x}_t, \mathbf{u}_t)dt\}$ Stochastic HJB equation: $\beta V^*(t,\mathbf{x}) - V_t^*(t,\mathbf{x}) = \min_{\mathbf{u}_t}\{L(\mathbf{x}, \mathbf{u}_t) +$ $V_x^{*T}(t,\mathbf{x})\mathbf{f}(\mathbf{x}, \mathbf{u}_t) + \frac{1}{2}Tr[V_{\mathbf{xx}}^*(t,\mathbf{x})\mathbf{B}\Sigma\mathbf{B}^T]\}$ Infinite horizon: $\Phi(t_f) = 0$ V^* is not function of time.
Deterministic System	Optimization Problem: $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, \mathbf{u}_n)$ $\min_{\mathbf{u}_{0\to N-1}} \{\alpha^N \Phi(N) + \sum_{k=0}^{N-1} \alpha^k L(\mathbf{x}_k, \mathbf{u}_k)\} \quad \alpha \in [0,1]$ Bellman equation: $V^*(n,\mathbf{x}) = \min_{u_n}\{L(\mathbf{x}, \mathbf{u}_n)$ $+ \alpha V^*(n+1, \mathbf{x}_{n+1})\}$ Infinite time horizon: $\alpha \in [0,1)$ $\Phi(N) = 0$ V^* is not function of time.	Optimization Problem: $d\mathbf{x} = \mathbf{f}(\mathbf{x}_{(t)}, \mathbf{u}_{(t)})dt$ $\min_{\mathbf{u}_{0\to t_f}} \{e^{-\beta t_f}\Phi(t_f) + \int_0^{t_f} e^{-\beta t} L(\mathbf{x}_t, \mathbf{u}_t)dt\}$ HJB equation: $\beta V^*(t,\mathbf{x}) - V_t^*(t,\mathbf{x}) = \min_{\mathbf{u}_{(t)}}\{L(\mathbf{x}_t, \mathbf{u}_t)$ $+ V_{\mathbf{x}}^{*T}(t,\mathbf{x})\mathbf{f}(\mathbf{x}, \mathbf{u})\}$ Infinite time horizon: $\Phi(t_f) = 0$ V^* is not function of time.

Section 1.3.3

Section 1.3.1

$\mathbf{w}_n \sim P_{\mathbf{w}}(\cdot \mid \mathbf{x}_n, \mathbf{u}_n) = \delta(\mathbf{w})$

$\mathbf{w}_t \sim \mathcal{N}(0, 0)$ i.e. $\Sigma = \mathbf{0}$

Solving Discrete Time LQG

- Using the appropriate Bellman equation

$$V^*(n, \mathbf{x}) = \min_{\mathbf{u}_n} \left[L_n(\mathbf{x}, \mathbf{u}_n) + \alpha E_{\mathbf{x}' \sim P_f(\cdot|\mathbf{x}, \mathbf{u}_n)} \left[V^*\left(n+1, \mathbf{x}'\right) \right] \right]$$

- Like the LQR case, we need an Ansatz for the value function

$$V^*(n, \mathbf{x}) = \frac{1}{2}\mathbf{x}^T \mathbf{S}_n \mathbf{x} + \upsilon_n$$

It accounts for the introduced process noise.

Solving Discrete Time LQG

- Plugging in the Ansatz and the cost function

$$V^*(n, \mathbf{x}) = \min_{\mathbf{u}_n} \left[L_n(\mathbf{x}, \mathbf{u}_n) + \alpha E_{\mathbf{x}' \sim P_f(\cdot|\mathbf{x}, \mathbf{u}_n)} \left[V^* \left(n+1, \mathbf{x}' \right) \right] \right]$$

$$V^*(n, \mathbf{x}) = \min_{\mathbf{u}_n} \frac{1}{2} E \left[\mathbf{x}_n^T \mathbf{Q}_n \mathbf{x}_n + 2\mathbf{u}_n^T \mathbf{P}_n \mathbf{x}_n + \mathbf{u}_n^T \mathbf{R}_n \mathbf{u}_n + \alpha \mathbf{x}_{n+1}^T \mathbf{S}_{n+1} \mathbf{x}_{n+1} + 2\alpha v_{n+1} \right]$$

Solving Discrete Time LQG

- Plugging in the system dynamics and the assumption about zero mean noise.

$$V^*(n,\mathbf{x}) = \min_{\mathbf{u}_n} \frac{1}{2} E\left[\mathbf{x}_n^T \mathbf{Q}_n \mathbf{x}_n + 2\mathbf{u}_n^T \mathbf{P}_n \mathbf{x}_n + \mathbf{u}_n^T \mathbf{R}_n \mathbf{u}_n + \alpha \mathbf{x}_{n+1}^T \mathbf{S}_{n+1} \mathbf{x}_{n+1} + 2\alpha v_{n+1}\right]$$

$$\mathbf{x}_{n+1} = \mathbf{A}_n \mathbf{x}_n + \mathbf{B}_n \mathbf{u}_n + \mathbf{C}_n \mathbf{w}_n$$

$$V^*(n,\mathbf{x}) = \min_{\mathbf{u}_n} \frac{1}{2} E[\mathbf{x}_n^T \mathbf{Q}_n \mathbf{x}_n + 2\mathbf{u}_n^T \mathbf{P}_n \mathbf{x}_n + \mathbf{u}_n^T \mathbf{R}_n \mathbf{u}_n + \alpha (\mathbf{A}_n \mathbf{x}_n + \mathbf{B}_n \mathbf{u}_n)^T \mathbf{S}_{n+1} (\mathbf{A}_n \mathbf{x}_n + \mathbf{B}_n \mathbf{u}_n)$$
$$+ \alpha \mathbf{w}_n^T \mathbf{C}_n^T \mathbf{S}_{n+1} \mathbf{C}_n \mathbf{w}_n + 2\alpha v_{n+1}]$$

Notice that the linear term with respect to w_n has vanished!

Solving Discrete Time LQG

- Using $Tr(\mathbf{AB}) = Tr(\mathbf{BA})$ & $E[\mathbf{w}_n \mathbf{w}_n^T] = \mathbf{I}$

$$V^*(n, \mathbf{x}) = \min_{\mathbf{u}_n} \frac{1}{2} E[\mathbf{x}_n^T \mathbf{Q}_n \mathbf{x}_n + 2\mathbf{u}_n^T \mathbf{P}_n \mathbf{x}_n + \mathbf{u}_n^T \mathbf{R}_n \mathbf{u}_n + \alpha(\mathbf{A}_n \mathbf{x}_n + \mathbf{B}_n \mathbf{u}_n)^T \mathbf{S}_{n+1} (\mathbf{A}_n \mathbf{x}_n + \mathbf{B}_n \mathbf{u}_n) + \alpha \mathbf{w}_n^T \mathbf{C}_n^T \mathbf{S}_{n+1} \mathbf{C}_n \mathbf{w}_n + 2\alpha v_{n+1}]$$

The only stochastic term

$$V^*(n, \mathbf{x}) = \min_{\mathbf{u}_n} \frac{1}{2} \{\mathbf{x}_n^T \mathbf{Q}_n \mathbf{x}_n + 2\mathbf{u}_n^T \mathbf{P}_n \mathbf{x}_n + \mathbf{u}_n^T \mathbf{R}_n \mathbf{u}_n + \alpha(\mathbf{A}_n \mathbf{x}_n + \mathbf{B}_n \mathbf{u}_n)^T \mathbf{S}_{n+1} (\mathbf{A}_n \mathbf{x}_n + \mathbf{B}_n \mathbf{u}_n) + \alpha Tr(\mathbf{S}_{n+1} \mathbf{C}_n \mathbf{C}_n^T) + 2\alpha v_{n+1}\}$$

Note: there is no expectation anymore!

Solving Discrete Time LQG

- Minimizing the RHS w.r.t the control input by setting the gradient to zero

$$V^*(n,\mathbf{x}) = \min_{\mathbf{u}_n} \frac{1}{2}\{\mathbf{x}_n^T\mathbf{Q}_n\mathbf{x}_n + 2\mathbf{u}_n^T\mathbf{P}_n\mathbf{x}_n + \mathbf{u}_n^T\mathbf{R}_n\mathbf{u}_n + \alpha(\mathbf{A}_n\mathbf{x}_n + \mathbf{B}_n\mathbf{u}_n)^T\mathbf{S}_{n+1}(\mathbf{A}_n\mathbf{x}_n + \mathbf{B}_n\mathbf{u}_n)$$
$$+ \alpha Tr(\mathbf{S}_{n+1}\mathbf{C}_n\mathbf{C}_n^T) + 2\alpha v_{n+1}\}$$

$$\mathbf{u}^*(n,\mathbf{x}) = -(\mathbf{R}_n + \alpha\mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{B}_n)^{-1}(\mathbf{P}_n + \alpha\mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{A}_n)\mathbf{x}$$

This found the optimal control. However we still need to determine S_n

Solving Discrete Time LQG

- Substituting the optimal control in the equation

$$V^*(n,\mathbf{x}) = \frac{1}{2}\mathbf{x}_n^T\Big[\mathbf{Q}_n + \alpha\mathbf{A}_n^T\mathbf{S}_{n+1}\mathbf{A}_n - (\mathbf{P}_n + \alpha\mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{A}_n)^T(\mathbf{R}_n + \alpha\mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{B}_n)^{-1}$$
$$(\mathbf{P}_n + \alpha\mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{A}_n)\Big]\mathbf{x}_n + \alpha\Big[\frac{1}{2}Tr(\mathbf{S}_{n+1}\mathbf{C}_n\mathbf{C}_n^T) + v_{n+1}\Big]$$

- Re-using the Ansatz for the LHS and then gathering and grouping the terms on RHS.

Solving Discrete Time LQG

$$\frac{1}{2}\mathbf{x}_n^T\Big[\mathbf{Q}_n + \alpha\mathbf{A}_n^T\mathbf{S}_{n+1}\mathbf{A}_n - (\mathbf{P}_n + \alpha\mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{A}_n)^T(\mathbf{R}_n + \alpha\mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{B}_n)^{-1}$$

$$(\mathbf{P}_n + \alpha\mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{A}_n) - \mathbf{S}_n\Big]\mathbf{x}_n + \Big[\frac{1}{2}\alpha Tr(\mathbf{S}_{n+1}\mathbf{C}_n\mathbf{C}_n^T) + \alpha v_{n+1} - v_n\Big] = 0.$$

The green term is a function of state vector to the power of 2.

The blue term is a function of state vector to the power of 0.

This equality should hold for all the values of state vector. Therefore the coefficients of state vector should be set to zero!

Solving Discrete Time LQG

$$\mathbf{S}_n = \mathbf{Q}_n + \alpha \mathbf{A}_n^T \mathbf{S}_{n+1} \mathbf{A}_n - (\mathbf{P}_n + \alpha \mathbf{B}_n^T \mathbf{S}_{n+1} \mathbf{A}_n)^T (\mathbf{R}_n + \alpha \mathbf{B}_n^T \mathbf{S}_{n+1} \mathbf{B}_n)^{-1} (\mathbf{P}_n + \alpha \mathbf{B}_n^T \mathbf{S}_{n+1} \mathbf{A}_n)$$

$$\upsilon_n = \frac{1}{2} \alpha Tr(\mathbf{S}_{n+1} \mathbf{C}_n \mathbf{C}_n^T) + \alpha \upsilon_{n+1}, \qquad \mathbf{S}_N = \mathbf{Q}_N \quad \upsilon_N = 0$$

If we choose α=1, this equation is identical to the one for the LQR case which is called discrete time Riccati equation.

These are difference equations with final values. Therefore they should be solved backward (in time)!

LQG Summary

- Value function

$$V^*(n,\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{S}_n\mathbf{x} + v_n$$

- Optimal control

cf. certainty-equivalence principle

$$\mathbf{u}^*(n,\mathbf{x}) = -(\mathbf{R}_n + \alpha\mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{B}_n)^{-1}(\mathbf{P}_n + \alpha\mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{A}_n)\mathbf{x}$$

For $\alpha=1$, exactly the same as LQR

- Riccati equation

$$\mathbf{S}_n = \mathbf{Q}_n + \alpha\mathbf{A}_n^T\mathbf{S}_{n+1}\mathbf{A}_n - (\mathbf{P}_n + \alpha\mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{A}_n)^T(\mathbf{R}_n + \alpha\mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{B}_n)^{-1}(\mathbf{P}_n + \alpha\mathbf{B}_n^T\mathbf{S}_{n+1}\mathbf{A}_n)$$

$$v_n = \frac{1}{2}\alpha Tr(\mathbf{S}_{n+1}\mathbf{C}_n\mathbf{C}_n^T) + \alpha v_{n+1}, \qquad \mathbf{S}_N = \mathbf{Q}_N \quad v_N = 0$$

LQG Summary

- Value function

$$V^*(n, \mathbf{x}) = \frac{1}{2}\mathbf{x}^T \mathbf{S}_n \mathbf{x} + \upsilon_n$$

The difference between LQR and LQG
"The value function in LQG is always greater than LQR"

- Optimal control

$$\mathbf{u}^*(n, \mathbf{x}) = -(\mathbf{R}_n + \alpha \mathbf{B}_n^T \mathbf{S}_{n+1} \mathbf{B}_n)^{-1}(\mathbf{P}_n + \alpha \mathbf{B}_n^T \mathbf{S}_{n+1} \mathbf{A}_n)\mathbf{x}$$

- Riccati equation

$$\mathbf{S}_n = \mathbf{Q}_n + \alpha \mathbf{A}_n^T \mathbf{S}_{n+1} \mathbf{A}_n - (\mathbf{P}_n + \alpha \mathbf{B}_n^T \mathbf{S}_{n+1} \mathbf{A}_n)^T (\mathbf{R}_n + \alpha \mathbf{B}_n^T \mathbf{S}_{n+1} \mathbf{B}_n)^{-1} (\mathbf{P}_n + \alpha \mathbf{B}_n^T \mathbf{S}_{n+1} \mathbf{A}_n)$$

$$\upsilon_n = \frac{1}{2}\alpha Tr(\mathbf{S}_{n+1} \mathbf{C}_n \mathbf{C}_n^T) + \alpha \upsilon_{n+1}, \qquad \mathbf{S}_N = \mathbf{Q}_N \quad \upsilon_N = 0$$

Proof

- We just need to prove that ν is nonnegative $V^*(n, \mathbf{x}) = \frac{1}{2}\mathbf{x}^T \mathbf{S}_n \mathbf{x} + \upsilon_n$
- We will prove it by induction: First we show that the base case (n = N) is correct. Then we show that if the assumption for n+1 holds, n should also be correct.

Base case: It is obvious because υ_N is equal to zero.

Induction: From equation (1.118), we realize that if υ_{n+1} is nonnegative, υ_n will be nonnegative if and only if $Tr(\mathbf{S}_{n+1}\mathbf{C}_n\mathbf{C}_n^T) \geq 0$. Now we will show this.

$$\upsilon_n = \frac{1}{2}\alpha Tr(\mathbf{S}_{n+1}\mathbf{C}_n\mathbf{C}_n^T) + \alpha\upsilon_{n+1},$$

$$\begin{aligned} Tr(\mathbf{S}_{n+1}\mathbf{C}_n\mathbf{C}_n^T) &= Tr(\mathbf{C}_n^T\mathbf{S}_{n+1}\mathbf{C}_n) \\ &= \sum_i \mathbf{C}^{iT}_n \mathbf{S}_{n+1} \mathbf{C}^i_n \end{aligned}$$

where $\mathbf{C}^i_n$ is a the *i*th column of $\mathbf{C}_n$. Since S_n is always positive semidefinite, $\mathbf{C}^{iT}_n \mathbf{S}_{n+1} \mathbf{C}^i_n \geq 0$ holds for all *i*. Therefore $Tr(\mathbf{S}_{n+1}\mathbf{C}_n\mathbf{C}_n^T) \geq 0$, and υ_n is always nonnegative.

Discrete time LQG: Infinite Horizon

Problem Definition

- Cost function

$$J = \frac{1}{2} E \left\{ \sum_{n=0}^{\infty} \alpha^n \begin{bmatrix} \mathbf{x}_n^T & \mathbf{u}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{P}^T \\ \mathbf{P} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{x}_n \\ \mathbf{u}_n \end{bmatrix} \right\}$$

- The summation upper limit is changed to infinity
- There is no terminal cost

- System dynamics

$$\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n + \mathbf{B}\mathbf{u}_n + \mathbf{C}\mathbf{w}_n \qquad \mathbf{x}(0) = \mathbf{x}_0 \text{ given}$$

All the coefficients are time invariant!

Solution

- Value function

$$V^*(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{S}\mathbf{x} + \upsilon$$

> • Using the same Ansatz for the value function.
> • Considering that the value function is not a function of time.

- Optimal controller

$$\mathbf{u}^*(\mathbf{x}) = -(\mathbf{R} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{B})^{-1}(\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})\mathbf{x}$$

- Discrete-time algebraic Riccati equation

$$\mathbf{S} = \mathbf{Q} + \alpha\mathbf{A}^T\mathbf{S}\mathbf{A} - (\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})^T(\mathbf{R} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{B})^{-1}(\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})$$

$$\upsilon = \frac{\alpha}{2(1-\alpha)}Tr(\mathbf{S}\mathbf{C}\mathbf{C}^T)$$

Solution

- Value function

$$V^*(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{S}\mathbf{x} + \upsilon$$

- Optimal controller

$$\mathbf{u}^*(\mathbf{x}) = -(\mathbf{R} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{B})^{-1}(\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})\mathbf{x}$$

• The optimal control is only a function of state vector.
• It is a linear feedback controller.

- Discrete-time algebraic Riccati equation

$$\mathbf{S} = \mathbf{Q} + \alpha\mathbf{A}^T\mathbf{S}\mathbf{A} - (\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})^T(\mathbf{R} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{B})^{-1}(\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})$$

$$\upsilon = \frac{\alpha}{2(1-\alpha)}Tr(\mathbf{S}\mathbf{C}\mathbf{C}^T)$$

Solution

- Value function

$$V^*(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{S}\mathbf{x} + v$$

- Optimal controller

$$\mathbf{u}^*(\mathbf{x}) = -(\mathbf{R} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{B})^{-1}(\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})\mathbf{x}$$

- Discrete-time algebraic Riccati equation

$$\mathbf{S} = \mathbf{Q} + \alpha\mathbf{A}^T\mathbf{S}\mathbf{A} - (\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})^T(\mathbf{R} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{B})^{-1}(\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})$$

$$v = \frac{\alpha}{2(1-\alpha)}Tr(\mathbf{S}\mathbf{C}\mathbf{C}^T)$$

• Using the result from the finite horizon problem.
• Considering that S is not a function of time.

Solution

- Value function

$$V^*(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{S}\mathbf{x} + \upsilon$$

- Optimal controller

$$\mathbf{u}^*(\mathbf{x}) = -(\mathbf{R} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{B})^{-1}(\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})\mathbf{x}$$

- Discrete-time algebraic Riccati equation

$$\mathbf{S} = \mathbf{Q} + \alpha\mathbf{A}^T\mathbf{S}\mathbf{A} - (\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})^T(\mathbf{R} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{B})^{-1}(\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})$$

$$\upsilon = \frac{\alpha}{2(1-\alpha)}Tr(\mathbf{S}\mathbf{C}\mathbf{C}^T)$$

• It accounts for the stochasticity of the problem.
• It is always nonnegative.

Solution

- Value function

$$V^*(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{S}\mathbf{x} + v$$

- Optimal controller

$$\mathbf{u}^*(\mathbf{x}) = -(\mathbf{R} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{B})^{-1}(\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})\mathbf{x}$$

- Discrete-time algebraic Riccati equation

$$\mathbf{S} = \mathbf{Q} + \alpha\mathbf{A}^T\mathbf{S}\mathbf{A} - (\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})^T(\mathbf{R} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{B})^{-1}(\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})$$

$$v = \frac{\alpha}{2(1-\alpha)}Tr(\mathbf{S}\mathbf{C}\mathbf{C}^T)$$

It is the same equation as the LQR case, if $\alpha = 1$.

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Solution

- Value function

$$V^*(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{S}\mathbf{x} + v$$

- Optimal controller

$$\mathbf{u}^*(\mathbf{x}) = -(\mathbf{R} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{B})^{-1}(\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})\mathbf{x}$$

- Discrete-time algebraic Riccati equation

$$\mathbf{S} = \mathbf{Q} + \alpha\mathbf{A}^T\mathbf{S}\mathbf{A} - (\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})^T(\mathbf{R} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{B})^{-1}(\mathbf{P} + \alpha\mathbf{B}^T\mathbf{S}\mathbf{A})$$

$$v = \frac{\alpha}{2(1-\alpha)}Tr(\mathbf{S}\mathbf{C}\mathbf{C}^T)$$

$\alpha \neq 1$
However it can approach 1 in the limit!

Continuous Time LQG: Finite Horizon

Problem Definition

- Linear system dynamics contaminated by a Gaussian noise.

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{C}(t)\mathbf{w}(t), \quad \mathbf{x}(0) = x_0$$

Problem Definition

- Linear system dynamics contaminated by a Gaussian noise

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{C}(t)\mathbf{w}(t), \quad \mathbf{x}(0) = x_0$$

$$E[\mathbf{w}(t)] = \mathbf{0}$$

Zero mean

$$E[\mathbf{w}(t)\mathbf{w}(\tau)^T] = \mathbf{I}\delta(t-\tau)$$

Uncorrelated

Gaussian process should be defined by its:
Mean & **Covariance**

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Problem Definition

- Linear system dynamics contaminated by a Gaussian noise

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{C}(t)\mathbf{w}(t), \quad \mathbf{x}(0) = x_0$$

Can be any arbitrary functions of time.

However they should be continuous.

Problem Definition

- Pure quadratic cost function

$$J = \frac{1}{2} E \left\{ e^{-\beta T} \mathbf{x}^T(T) \mathbf{Q}_T \mathbf{x}(T) + \int_0^T e^{-\beta t} \begin{bmatrix} \mathbf{x}^T(t) & \mathbf{u}^T(t) \end{bmatrix} \begin{bmatrix} \mathbf{Q}(t) & \mathbf{P}^T(t) \\ \mathbf{P}(t) & \mathbf{R}(t) \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} dt \right\}$$

- Without *E* and β it is equivalent to the LQR cost function.
- β is discount factor or decay factor.
- For finite horizon it is usually chosen 0.
- However for infinite horizon, it should be chosen greater than 0; otherwise the summation does not exist!

How to Solve it?

	Discrete Time	Continuous Time
Stochastic System	Optimization Problem: $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, \mathbf{u}_n) + \mathbf{w}_n$ $\mathbf{w}_n \sim P_{\mathbf{w}}(\cdot \mid \mathbf{x}_n, \mathbf{u}_n)$ $\min_{\mathbf{u}_{0 \to N-1}} E\{\alpha^N \Phi(N) + \sum_{k=0}^{N-1} \alpha^k L(\mathbf{x}_k, \mathbf{u}_k)\} \quad \alpha \in [0,1]$ Stochastic Bellman equation: $V^*(n, \mathbf{x}) = \min_{u_n}\{L(x, u_n) + \alpha E[V^*(n+1, \mathbf{x}_{n+1})]\}$ Infinite horizon: $\alpha \in [0,1)$ $\Phi(N) = 0$ V^* is not function of time.	Optimization Problem: $d\mathbf{x} = \mathbf{f}(\mathbf{x}_t, \mathbf{u}_t)dt + \mathbf{B}(\mathbf{x}_t, \mathbf{u}_t)d\mathbf{w}_t$ $\mathbf{w}_t \sim \mathcal{N}(0, \Sigma)$ $\min_{\mathbf{u}_{0 \to t_f}} E\{e^{-\beta t_f}\Phi(t_f) + \int_0^{t_f} e^{-\beta t} L(\mathbf{x}_t, \mathbf{u}_t)dt\}$ Stochastic HJB equation: $\beta V^*(t, \mathbf{x}) - V_t^*(t, \mathbf{x}) = \min_{\mathbf{u}_t}\{L(\mathbf{x}, \mathbf{u}_t) + V_x^{*T}(t, \mathbf{x})\mathbf{f}(\mathbf{x}, \mathbf{u}_t) + \frac{1}{2}Tr[V_{\mathbf{xx}}^*(t, \mathbf{x})\mathbf{B}\Sigma\mathbf{B}^T]\}$ Infinite horizon: $\Phi(t_f) = 0$ V^* is not function of time.
Deterministic System	Optimization Problem: $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n, \mathbf{u}_n)$ $\min_{\mathbf{u}_{0 \to N-1}} \{\alpha^N \Phi(N) + \sum_{k=0}^{N-1} \alpha^k L(\mathbf{x}_k, \mathbf{u}_k)\} \quad \alpha \in [0,1]$ Bellman equation: $V^*(n, \mathbf{x}) = \min_{u_n}\{L(\mathbf{x}, \mathbf{u}_n) + \alpha V^*(n+1, \mathbf{x}_{n+1})\}$ Infinite time horizon: $\alpha \in [0,1)$ $\Phi(N) = 0$ V^* is not function of time.	Optimization Problem: $d\mathbf{x} = \mathbf{f}(\mathbf{x}_{(t)}, \mathbf{u}_{(t)})dt$ $\min_{\mathbf{u}_{0 \to t_f}} \{e^{-\beta t_f}\Phi(t_f) + \int_0^{t_f} e^{-\beta t} L(\mathbf{x}_t, \mathbf{u}_t)dt\}$ HJB equation: $\beta V^*(t, \mathbf{x}) - V_t^*(t, \mathbf{x}) = \min_{\mathbf{u}_{(t)}}\{L(\mathbf{x}_t, \mathbf{u}_t) + V_{\mathbf{x}}^{*T}(t, \mathbf{x})\mathbf{f}(\mathbf{x}, \mathbf{u})\}$ Infinite time horizon: $\Phi(t_f) = 0$ V^* is not function of time.

Section 1.3.3

Section 1.3.1

$\mathbf{w}_n \sim P_{\mathbf{w}}(\cdot \mid \mathbf{x}_n, \mathbf{u}_n) = \delta(\mathbf{w})$

$\mathbf{w}_t \sim \mathcal{N}(0, 0)$ i.e. $\Sigma = 0$

Solving Continuous Time LQG

- Using the appropriate HJB equation

$$\beta V^*(t,\mathbf{x}) - V_t^*(t,\mathbf{x}) = \min_{\mathbf{u}(t)} \left\{ L(\mathbf{x},\mathbf{u}(t)) + V_{\mathbf{x}}^{*T}\mathbf{f}_t(\mathbf{x},\mathbf{u}(t)) + \frac{1}{2}\mathrm{Tr}[V_{\mathbf{xx}}^*\mathbf{C}(t)\mathbf{C}^T(t)] \right\}$$

- Like the LQR case, we need an Ansatz for value function

$$V^*(t,\mathbf{x}) = \frac{1}{2}\mathbf{x}^T(t)\mathbf{S}(t)\mathbf{x}(t) + v(t)$$

It accounts for the introduced process noise.

Solving Continuous Time LQG

- HJB equation

$$\beta V^*(t,\mathbf{x}) - V_t^*(t,\mathbf{x}) = \min_{\mathbf{u}(t)} \left\{ L(\mathbf{x},\mathbf{u}(t)) + V_{\mathbf{x}}^{*T}\mathbf{f}_t(\mathbf{x},\mathbf{u}(t)) + \frac{1}{2}\mathrm{Tr}[V_{\mathbf{xx}}^*\mathbf{C}(t)\mathbf{C}^T(t)] \right\}$$

- Required partial derivatives

$$V^*(t,\mathbf{x}) = \frac{1}{2}\mathbf{x}^T(t)\mathbf{S}(t)\mathbf{x}(t) + v(t)$$

$$V_t^*(t,\mathbf{x}) = \frac{1}{2}\mathbf{x}^T(t)\dot{\mathbf{S}}(t)\mathbf{x}(t) + \dot{v}(t)$$

$$V_{\mathbf{x}}^*(t,\mathbf{x}) = \mathbf{S}(t)\mathbf{x}(t)$$

$$V_{\mathbf{xx}}^*(t,\mathbf{x}) = \mathbf{S}(t)$$

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Solving Continuous Time LQG

- Plugging in the Ansatz, system dynamics, and the cost function

$$\beta V^*(t,\mathbf{x}) - V_t^*(t,\mathbf{x}) = \min_{\mathbf{u}(t)} \left\{ L(\mathbf{x},\mathbf{u}(t)) + V_{\mathbf{x}}^{*T}\mathbf{f}_t(\mathbf{x},\mathbf{u}(t)) + \frac{1}{2}\mathrm{Tr}[V_{\mathbf{xx}}^*\mathbf{C}(t)\mathbf{C}^T(t)] \right\}$$

$$\beta V^* - V_t^* = \min_{\mathbf{u}} \frac{1}{2}\{\mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{u}^T\mathbf{P}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u} + 2\mathbf{x}^T\mathbf{S}(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) + Tr(\mathbf{S}\mathbf{C}\mathbf{C}^T)\}$$

Solving Continuous Time LQG

- Minimizing the RHS w.r.t the control input by setting the gradient to zero

$$\mathbf{u}^*(t,\mathbf{x}) = -\mathbf{R}(t)^{-1}\left(\mathbf{P}(t) + \mathbf{B}^T(t)\mathbf{S}(t)\right)\mathbf{x}$$

- Substituting optimal control in the equation and gathering and grouping terms on RHS

$$\frac{1}{2}\mathbf{x}^T\Big[\mathbf{S}(t)\mathbf{A}(t) + \mathbf{A}^T(t)\mathbf{S}(t) - \left(\mathbf{P}(t) + \mathbf{B}^T(t)\mathbf{S}(t)\right)^T\mathbf{R}^{-1}\left(\mathbf{P}(t) + \mathbf{B}^T(t)\mathbf{S}(t)\right)$$
$$+ \mathbf{Q}(t) + \dot{\mathbf{S}}(t) - \beta\mathbf{S}\Big]\mathbf{x} + \Big[\dot{\upsilon}(t) - \beta\upsilon(t) + \frac{1}{2}Tr(\mathbf{S}\mathbf{C}\mathbf{C}^T)\Big] = 0$$

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The green term is a function of state vector to the power of 2.

The blue term is a function of state vector to the power of 0.

This equality should hold for all the values of state vector.
Therefore the coefficients of state vector should be set to zero!

LQG Summary

- Value function

$$V^*(t,\mathbf{x}) = \frac{1}{2}\mathbf{x}^T(t)\mathbf{S}(t)\mathbf{x}(t) + v(t)$$

The difference between LQR and LQG
"The value function in LQG is always greater than LQR"

- Optimal control

$$\mathbf{u}^*(t,\mathbf{x}) = -\mathbf{R}(t)^{-1}\left(\mathbf{P}(t) + \mathbf{B}^T(t)\mathbf{S}(t)\right)\mathbf{x}$$

- Riccati equation

For β=0,
Exactly the same as LQR

$$\dot{\mathbf{S}} = \beta\mathbf{S} - \mathbf{S}\mathbf{A} - \mathbf{A}^T\mathbf{S} + \left(\mathbf{P} + \mathbf{B}^T\mathbf{S}\right)^T\mathbf{R}^{-1}\left(\mathbf{P} + \mathbf{B}^T\mathbf{S}\right) - \mathbf{Q}, \quad \text{with } \mathbf{S}(T) = \mathbf{Q}_T.$$

$$\dot{v} = \beta v(t) - \frac{1}{2}Tr(\mathbf{S}\mathbf{C}\mathbf{C}^T) \quad \text{with } v(T) = 0.$$

Continuous-time LQG: Infinite Horizon

Problem Definition

- The summation upper limit is changed to infinity.
- There is no terminal cost.

- Cost function

$$J = \frac{1}{2} E \left\{ \int_0^{\infty} e^{-\beta t} \begin{bmatrix} \mathbf{x}^T(t) & \mathbf{u}^T(t) \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{P}^T \\ \mathbf{P} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{bmatrix} dt \right\}$$

- System dynamics

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{C}\mathbf{w}(t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \text{ given}$$

Problem Definition

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- The summation upper limit is changed to infinity.
- There is no terminal cost.

$\beta > 0 \quad \beta \neq 0$

It can approach in limit to 0.

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All the coefficients are time invariant!

Solution

- Value function

$$V^*(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{S}\mathbf{x} + \upsilon$$

- Optimal controller

$$\mathbf{u}^*(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{P} + \mathbf{B}^T\mathbf{S})\mathbf{x}$$

> • Using the same Ansatz for the value function.
> • Considering that the value function is not a function of time.

- Continuous-time algebraic Riccati equation

$$-\beta\mathbf{S} + \mathbf{S}\mathbf{A} + \mathbf{A}^T\mathbf{S} - \left(\mathbf{P} + \mathbf{B}^T\mathbf{S}\right)^T\mathbf{R}^{-1}\left(\mathbf{P} + \mathbf{B}^T\mathbf{S}\right) + \mathbf{Q} = 0$$

$$\upsilon(t) = \frac{1}{2\beta}Tr(\mathbf{S}\mathbf{C}\mathbf{C}^T)$$

Solution

- Value function

$$V^*(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{S}\mathbf{x} + \upsilon$$

- Optimal controller

$$\mathbf{u}^*(\underline{\mathbf{x}}) = -\mathbf{R}^{-1}(\mathbf{P} + \mathbf{B}^T\mathbf{S})\underline{\mathbf{x}}$$

- The optimal control is only a function of state vector.
- It is a linear feedback controller.

- Continuous-time algebraic Riccati equation

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$$\upsilon(t) = \frac{1}{2\beta}Tr(\mathbf{S}\mathbf{C}\mathbf{C}^T)$$

- Using the result from the finite horizon problem.
- Considering that S is not a function of time.

Solution

- Value function

$$V^*(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{S}\mathbf{x} + \upsilon$$

- Optimal controller

$$\mathbf{u}^*(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{P} + \mathbf{B}^T\mathbf{S})\mathbf{x}$$

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$$\upsilon(t) = \frac{1}{2\beta}Tr(\mathbf{S}\mathbf{C}\mathbf{C}^T)$$

- It accounts for the stochasticity of the problem.
- It is always nonnegative.

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Solution

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$$V^*(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{S}\mathbf{x} + \upsilon$$

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- Continuous-time algebraic Riccati equation

$$-\beta\mathbf{S} + \mathbf{S}\mathbf{A} + \mathbf{A}^T\mathbf{S} - (\mathbf{P} + \mathbf{B}^T\mathbf{S})^T\mathbf{R}^{-1}(\mathbf{P} + \mathbf{B}^T\mathbf{S}) + \mathbf{Q} = 0$$

$$\upsilon(t) = \frac{1}{2\beta}Tr(\mathbf{S}\mathbf{C}\mathbf{C}^T)$$

It is the same equation as the LQR case, if β =0.

Solution

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$$V^*(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{S}\mathbf{x} + \upsilon$$

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$$\mathbf{u}^*(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{P} + \mathbf{B}^T\mathbf{S})\mathbf{x}$$

- Continuous-time algebraic Riccati equation

$$-\beta\mathbf{S} + \mathbf{S}\mathbf{A} + \mathbf{A}^T\mathbf{S} - (\mathbf{P} + \mathbf{B}^T\mathbf{S})^T\mathbf{R}^{-1}(\mathbf{P} + \mathbf{B}^T\mathbf{S}) + \mathbf{Q} = 0$$

$$\upsilon(t) = \frac{1}{2\beta}Tr(\mathbf{S}\mathbf{C}\mathbf{C}^T)$$

$\beta \neq 0$
However it can be arbitrary small!

Practical considerations

Robustness

LQG works well for SISO

MIMO can be problematic in terms of robustness

Guaranteed Margins for LQG Regulators

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Abstract—There are none.

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